

# COMPOSITION OPERATORS ON THE DISCRETE ANALOGUE OF GENERALIZED HARDY SPACE ON HOMOGENOUS TREES

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**ABSTRACT.** In this paper, we study the basic properties such as boundedness and compactness of composition operators on discrete analogue of generalized Hardy space defined on a homogeneous rooted tree. Also, we compute the operator norm of composition operator when inducing symbol is automorphism of a homogenous tree.

## 1. INTRODUCTION

Let  $\Omega$  be a nonempty set and  $X$  be a complex Banach space of complex valued functions defined on  $\Omega$ . For a self map  $\phi$  of  $\Omega$ , the composition operator  $C_\phi$  induced by the symbol  $\phi$  is defined as

$$C_\phi(f) = g \text{ where } g(x) = f(\phi(x)) \text{ for all } x \in \Omega \text{ and } f \in X.$$

In the classical case,  $\Omega$  is the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and the choices for  $X$  are analytic functions spaces, eg. the Hardy space  $H^p$ , the Bergman space  $A^p$ , the Bloch space  $\mathcal{B}$ , etc. The study of composition operators on various analytic function spaces defined on  $\mathbb{D}$  is well known. There are excellent books on composition operators, see [10, 12, 13] and the references therein. The approach in the first two books [10, 12] are function theoretic whereas [13] deals in measure theoretic point of view. Also, there a number of articles dealing with composition operators on different transform spaces, see for example [1, 2, 7]. In this article,  $\Omega$  will be a homogeneous rooted tree and  $X$  the discrete analogue of generalized Hardy space introduced in [11].

In the recent years, there has been a great interest in studying operator theory on discrete structure such as graphs, in particular on an infinite tree graph [3, 4, 5, 8, 9]. In [6], Colonna et al. studied composition operators on Lipschitz functions on a tree with edge counting metric to the complex plane with Euclidean metric, which is a discrete analogue of Bloch space, because Bloch space is also consisting of only Lipschitz functions on the unit disk under Hyperbolic metric to the complex plane with Euclidean metric. In [11], the present authors defined discrete analogue ( $\mathbb{T}_p$ ) of generalized Hardy spaces on homogeneous rooted tree. In the same article multiplication operators on  $\mathbb{T}_p$  spaces are studied.

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In this article, we deal with the study of composition operators on  $\mathbb{T}_p$  spaces. We refer to Section 2 for the definitions of homogeneous rooted tree and  $\mathbb{T}_p$  spaces. In Section 3, we consider the boundedness of composition operators on  $\mathbb{T}_p$  spaces and some of its consequences including norm estimates. In Section 4, we consider the compactness of composition operators on  $\mathbb{T}_p$  spaces and derive equivalent conditions for compactness. Finally, in Section 5, we present three examples to show the following: there are self maps of  $T$  which do not induce bounded composition operator on  $\mathbb{T}_p$ ; there exists a bounded composition operator on  $\mathbb{T}_p$  which is not compact; there are unbounded self maps of  $T$  which induces compact composition operators on  $\mathbb{T}_p$  for the case of  $(q+1)$ -homogeneous trees with  $q \geq 2$ .

## 2. PRELIMINARIES AND LEMMAS

Let  $G = (V, E)$  be a graph such that  $E \subseteq V \times V$ , where the elements of the sets  $V$  and  $E$  are called vertices and edges of the graph  $G$ , respectively. We shall not always distinguish between a graph and its vertex set and so, we may write  $x \in G$  (rather than  $x \in V$ ) and by a function defined on a graph, we mean a function defined on its vertices. Two vertices  $x, y \in G$  are said to be *neighbours* (denoted by  $x \sim y$ ) if  $(x, y) \in E$ . If all the vertices of  $G$  have the same number  $k$  of neighbours, then the graph is said to be *k-homogeneous* or *k-regular* graph. A *finite path* is a nonempty subgraph  $P = (V, E)$  of the form  $V = \{x_0, x_1, \dots, x_k\}$  and  $E = \{(x_0, x_1), (x_1, x_2), \dots, (x_{k-1}, x_k)\}$ , where  $x_i$ 's are distinct. In this case, we call  $P$  be a path between  $x_0$  and  $x_k$ . If  $P$  is a path between  $x_0$  and  $x_k$  ( $k \geq 2$ ), then  $P$  with an additional edge  $(x_n, x_0)$  is called a *cycle*. A nonempty graph  $G$  is called *connected* if for any two of its vertices, there is a path between them. A connected graph without cycles is called a *tree*. Thus, any two vertices of a tree are linked by a unique path. The *distance* between any two vertex of a tree is the number of edges in the unique path connecting them. Sometimes it is convenient to consider one vertex of a tree as special; such a vertex is then called the root of this tree. A tree  $T$  with fixed root  $o$  is called a *rooted tree*. If  $G$  is a rooted tree with root  $o$ , then  $|v|$  denotes the distance between the root  $o$  and the vertex  $v$ . Further, the *parent* (denoted by  $v^-$ ) of a vertex  $v$ , which is not a root, is the unique vertex  $w \in G$  such that  $w \sim v$  and  $|w| = |v| - 1$ . In this case,  $v$  is called *child* of  $w$ . For basic issues regarding graph theory, one can refer standard texts on this subject.

Throughout the paper, unless otherwise stated explicitly,  $T$  denotes a homogeneous rooted tree (hence infinite graph),  $\phi$  denotes a self map of  $T$ ,  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

For  $p \in (0, \infty]$ , the *Hardy space*  $H^p$  consists of all those analytic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that  $\|f\|_p < \infty$ , where

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f)$$

and

$$M_p(r, f) = \begin{cases} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} & \text{if } p \in (0, \infty) \\ \sup_{|z|=r} |f(z)| & \text{if } p = \infty. \end{cases}$$

The *generalized Hardy space*  $H_g^p$  is defined similarly, upon replacing analytic functions by measurable functions.

As in [11], in a  $(q+1)$ -homogeneous tree  $T$  rooted at  $o$ , we define

$$\|f\|_p := \sup_{n \in \mathbb{N}_0} M_p(n, f),$$

where  $M_p(0, f) := |f(o)|$  and for every  $n \in \mathbb{N}$ ,

$$M_p(n, f) := \begin{cases} \left( \frac{1}{(q+1)q^{n-1}} \sum_{|v|=n} |f(v)|^p \right)^{\frac{1}{p}} & \text{if } p \in (0, \infty) \\ \max_{|v|=n} |f(v)| & \text{if } p = \infty. \end{cases}$$

The discrete analogue of the generalized Hardy space, denoted by  $\mathbb{T}_{q,p}$ , is then defined by

$$\mathbb{T}_{q,p} := \{f : T \rightarrow \mathbb{C} \mid \|f\|_p < \infty\}$$

for every  $p \in (0, \infty]$ . For the sake of simplicity, we shall write  $\mathbb{T}_{q,p}$  as  $\mathbb{T}_p$ . Throughout the discussion,  $\|\cdot\|$  denotes  $\|\cdot\|_p$  in  $\mathbb{T}_p$  spaces. The following results proved by the present authors [11] are needed for our present investigation.

**Lemma A.** *For  $1 \leq p \leq \infty$ ,  $\|\cdot\|_p$  induces a Banach space structure on the space  $\mathbb{T}_p$ .*

**Lemma B.** (Growth Estimate) *Let  $T$  be a  $(q+1)$ -homogeneous tree rooted at  $o$  and  $0 < p < \infty$ . Then, for  $v \in T$ , we have the following: If  $f \in \mathbb{T}_p$ , then*

$$|f(v)| \leq \{(q+1)q^{|v|-1}\}^{\frac{1}{p}} \|f\|_p.$$

**Lemma C.** *Norm convergence in  $\mathbb{T}_p$  implies pointwise convergence. That is,*

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0 \Rightarrow \lim_{n \rightarrow \infty} f_n(v) = f(v) \text{ for each } v \in T.$$

### 3. BOUNDED COMPOSITION OPERATORS

A linear operator  $A$  from a normed linear space  $X$  to a normed linear space  $Y$  is said to be *bounded* if the operator norm  $\|A\| = \sup\{\|Ax\|_Y : \|x\|_X = 1\}$  is finite.

Before we proceed to discuss our results, it is appropriate to recall some basic results about bounded composition operators in the classical setting. For example (see [10, Corollary 3.7]), every analytic self map  $\phi$  of  $\mathbb{D}$  induces bounded composition operator  $C_\phi$  on  $H^p$ ,  $1 \leq p < \infty$ . Moreover,

$$(3.1) \quad \|C_\phi\|^p \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|}.$$

It is also known that (see [10, Theorem 3.8]) equality holds in (3.1) for every inner function of  $\mathbb{D}$  (for example, for every automorphism of  $\mathbb{D}$ ). For the case  $p = \infty$ , it is easy to see that  $\|C_\phi\| = 1$  for every analytic self map  $\phi$  of  $\mathbb{D}$ .

Now, for our setting, we let  $\phi$  be a self map of  $(q+1)$ -homogeneous rooted tree  $T$ . For  $n \in \mathbb{N}_0$  and  $w \in T$ , let  $N_\phi(n, w)$  denote the number of pre-images of  $w$  for  $\phi$  in  $|v| = n$ . That is  $N_\phi(n, w)$  is the number of elements in  $\{\phi^{-1}(w)\} \cap \{|v| = n\}$ . For  $w \in T$ , we define the weight function  $W$  as follows:

$$(3.2) \quad W(w) := \begin{cases} (q+1)q^{|w|-1} & \text{if } w \in T \setminus \{o\} \\ 1 & \text{if } w = o. \end{cases}$$

Let  $|D_n|$  denote the number of vertices with  $|v| = n$ . Thus,

$$|D_n| = \begin{cases} (q+1)q^{n-1} & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0. \end{cases}$$

**Theorem 1.** *Every self map  $\phi$  of  $T$  induces bounded composition operator on  $\mathbb{T}_\infty$  with  $\|C_\phi\| = 1$ .*

*Proof.* For each  $f \in \mathbb{T}_\infty$  and every self map  $\phi$  of  $T$ , we have

$$\|C_\phi(f)\|_\infty = \|f \circ \phi\|_\infty = \sup_{w \in \phi(T)} |f(w)| \leq \|f\|_\infty.$$

Thus,  $C_\phi$  is bounded on  $\mathbb{T}_\infty$ . It is easy to see that  $\|\chi_{\{v\}} \circ \phi\|_\infty = 1$  for each  $v \in T$ , where  $\chi_{\{v\}}$  denotes the characteristic function on  $\{v\}$ . It gives that  $\|C_\phi\| = 1$ .  $\square$

In order to study the boundedness of the composition operators on  $\mathbb{T}_p$  for  $1 \leq p < \infty$ , it is convenient to deal with the case  $q = 1$  and  $q \geq 2$  independently. First, we begin with the case  $q = 1$ .

**Theorem 2.** *For every self map  $\phi$  of 2-homogeneous tree  $T$ ,  $C_\phi$  is bounded on  $\mathbb{T}_p$  with  $\|C_\phi\|^p \leq 2$ ,  $1 \leq p < \infty$ .*

*Proof.* By the growth estimate (Lemma B) for 2-homogeneous trees, it follows that  $|f(v)|^p \leq 2\|f\|^p$  for all  $v \in T$  and  $f \in \mathbb{T}_p$ . So,

$$M_p^p(0, C_\phi f) = |f(\phi(o))|^p \leq 2\|f\|^p.$$

Since  $|D_n| = 2$  for all  $n \in \mathbb{N}$ ,

$$M_p^p(n, C_\phi f) = \frac{1}{|D_n|} \sum_{|v|=n} |f(\phi(v))|^p \leq \frac{2\|f\|^p + 2\|f\|^p}{2} = 2\|f\|^p,$$

showing that  $\|C_\phi(f)\|^p \leq 2\|f\|^p$  and the result follows.  $\square$

**Theorem 3.** *If  $T$  is a  $(q+1)$ -homogeneous tree with  $q \geq 2$  such that*

$$(3.3) \quad \sup_{n \in \mathbb{N}} \left( \sum_{|v|=n} q^{|\phi(v)|-n} \right) < \infty,$$

*then  $C_\phi$  is bounded on  $\mathbb{T}_p$ ,  $1 \leq p < \infty$ .*

*Proof.* For  $n \in \mathbb{N}$ ,  $w \in T$  and  $f \in \mathbb{T}_p$ , by definition and Lemma B on growth estimate, we have

$$M_p^p(n, C_\phi f) \leq \frac{1}{|D_n|} \sum_{|v|=n} (q+1)q^{|\phi(v)|-1} \|f\|^p = \sum_{|v|=n} q^{|\phi(v)|-n} \|f\|^p.$$

Moreover,  $M_p^p(0, C_\phi f) = |f(\phi(o))|^p \leq (q+1)q^{|\phi(o)|-1} \|f\|^p$  and thus,

$$\|C_\phi f\|^p \leq \max \left\{ (q+1)q^{|\phi(o)|-1}, \sup_{n \in \mathbb{N}} \left( \sum_{|v|=n} q^{|\phi(v)|-n} \right) \right\} \|f\|^p$$

showing that  $C_\phi$  is bounded on  $\mathbb{T}_p$ .  $\square$

**Theorem 4.** *Let  $T$  be a  $(q+1)$ -homogeneous tree and  $1 \leq p < \infty$ . If  $C_\phi$  is bounded on  $\mathbb{T}_p$ , then*

$$\sup_{w \in T} \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_\phi(n, w) \right\} \leq \|C_\phi\|^p.$$

*Proof.* For each  $w \in T$ , define  $f_w = \{W(w)\chi_{\{w\}}\}^{\frac{1}{p}}$ , where  $W$  is defined in (3.2). It is easy to verify that for every  $w \in T$ ,  $M_p(n, f_w) = 1$  when  $n = |w|$  and 0 otherwise. This observation gives that  $\|f_w\| = 1$  for all  $w \in T$ . Now, for each fixed  $w \in T$ , we have for  $n \in \mathbb{N}_0$ ,

$$\begin{aligned} M_p^p(n, C_\phi f_w) &= \frac{1}{|D_n|} \sum_{|v|=n} W(w)\chi_{\{w\}}(\phi(v)) \\ &= \frac{1}{|D_n|} \sum_{\substack{|v|=n \\ \phi(v)=w}} W(w) = \frac{W(w)}{|D_n|} N_\phi(n, w) \end{aligned}$$

which yields that

$$\|C_\phi f_w\|^p = \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_\phi(n, w) \right\}.$$

Consequently,

$$\|C_\phi\|^p = \sup_{\|f\|=1} \|C_\phi(f)\|^p \geq \sup_{w \in T} \|C_\phi(f_w)\|^p = \sup_{w \in T} \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_\phi(n, w) \right\}$$

and the desired conclusion follows.  $\square$

**Corollary 1.** *If  $C_\phi$  is bounded on  $\mathbb{T}_p$ , then*

$$\sup \{q^{|w|-n} N_\phi(n, w) : w \in T \setminus \{o\}, n \in \mathbb{N}\}$$

*is finite.*

*Proof.* For  $w \in T \setminus \{o\}$  and  $n \in \mathbb{N}$ , we note that

$$\frac{W(w)}{|D_n|} = q^{|w|-n}.$$

The desired result follows by Theorem 4.  $\square$

**Corollary 2.** *If  $\phi$  fixes the root, namely,  $\phi(o) = o$ , then  $\|C_\phi\| \geq 1$ .*

*Proof.* Let  $f$  be the characteristic function on the root  $o$ . Clearly,  $\|f\| = 1$  and  $M_p(0, C_\phi f) = |f(\phi(o))| = |f(o)| = 1$ . We see that

$$\|C_\phi\| = \sup_{\|g\|=1} \|C_\phi(g)\| \geq \|C_\phi(f)\| \geq M_p(0, C_\phi f) = 1$$

and the result follows.  $\square$

**Corollary 3.** *If  $\phi$  does not fix the root, i.e.  $\phi(o) \neq o$ , then*

$$\|C_\phi\|^p \geq (q+1)q^{|\phi(o)|-1}.$$

*Proof.* Let  $w = \phi(o)$  and, as before, consider  $f_w = \{W(w)\chi_{\{w\}}\}^{\frac{1}{p}}$ . Now, we observe that

$$\|f_w\| = 1 \quad \text{and} \quad M_p^p(0, C_\phi f_w) = |f_w(\phi(o))|^p = (q+1)q^{|w|-1}$$

which shows that  $\|C_\phi(f_w)\|^p \geq (q+1)q^{|w|-1}$  and the proof follows.  $\square$

**Corollary 4.** *If  $T$  is a 2-homogeneous tree and  $\phi(o) \neq o$ , then  $\|C_\phi\|^p = 2$ .*

*Proof.* Setting  $q = 1$  in Corollary 3 and Theorem 2 gives  $\|C_\phi\|^p \geq 2$  and  $\|C_\phi\|^p \leq 2$ , respectively.  $\square$

A self map  $\phi$  of  $T$  is called an automorphism of  $T$ , denoted as  $\phi \in \text{Aut}(T)$ , if  $\phi$  is bijective and any two vertices  $v, w$  are neighbours ( $v \sim w$ ) if and only if  $\phi(v) \sim \phi(w)$ . Now we will compute the norm of the composition operator  $C_\phi$  when the inducing symbol  $\phi$  is an automorphism of  $T$ .

**Theorem 5.** *Let  $T$  be a  $(q+1)$ -homogeneous tree and consider  $C_\phi$  on  $\mathbb{T}_p$ , where  $1 \leq p < \infty$  and  $\phi \in \text{Aut}(T)$ . Then we have*

- (i)  $\|C_\phi\| = 1$  if  $\phi(o) = o$
- (ii)  $\|C_\phi\|^p = (q+1)q^{|\phi(o)|-1}$  if  $\phi(o) \neq o$ .

*In particular, every  $\phi \in \text{Aut}(T)$  induces bounded composition operator  $C_\phi$  on  $\mathbb{T}_p$ .*

*Proof.* Let  $D_n = \{v \in T : |v| = n\}$  and consider the case  $\phi(o) = o$ . Then, for each  $n$ ,  $\phi$  is a bijective map from  $D_n$  to  $D_n$  (since  $\phi \in \text{Aut}(T)$  and  $\phi(o) = o$ ). For  $n \in \mathbb{N}_0$  and  $f \in \mathbb{T}_p$ , we thus have

$$M_p^p(n, C_\phi f) = \frac{1}{|D_n|} \sum_{|\phi(v)|=n} |f(\phi(v))|^p = M_p^p(n, f).$$

Taking supremum on both sides, we get  $\|C_\phi(f)\| = \|f\|$  which proves the first part.

Next, we consider the case  $\phi(o) \neq o$ . The result is obviously true for  $q = 1$ , by Corollary 4. Thus, it suffices to prove the theorem for  $(q + 1)$ -homogeneous tree with  $q \geq 2$ . Let  $k = |\phi(o)|$ . Since  $\phi \in \text{Aut}(T)$ , is easy to see that

Domain	Range of $\phi$ contained in	Number of circles
$D_0$	$D_k$	1
$D_m$ ( $1 \leq m \leq k - 1$ )	$D_{k+m}, D_{k+m-2}, \dots, D_{k-m}$	$m + 1$
$D_k$	$D_{2k}, D_{2k-2}, \dots, D_2, D_0$	$k + 1$
$D_{k+m+1}$ ( $m \geq 0$ )	$D_{2k+m+1}, D_{2k+m-1}, \dots, D_{2m+1}$	$k + 1$

$$M_p^p(0, C_\phi f) = |f(\phi(0))|^p \leq (q + 1)q^{k-1} \|f\|^p.$$

For the remaining part of the proof, we need to deal with the cases  $n = m$  ( $1 \leq m \leq k - 1$ ),  $n = k$ , and  $n \geq k + 1$  separately. We begin with

$$M_p^p(m, C_\phi f)$$

$$\begin{aligned}
&= \frac{1}{(q + 1)q^{m-1}} \sum_{|v|=m} |f(\phi(v))|^p \\
&\leq \frac{1}{(q + 1)q^{m-1}} \left\{ \sum_{|v|=k+m} |f(v)|^p + \sum_{|v|=k+m-2} |f(v)|^p + \dots + \sum_{|v|=k-m} |f(v)|^p \right\} \\
&\leq \frac{1}{(q + 1)q^{m-1}} \{ (q + 1)q^{k+m-1} + (q + 1)q^{k+m-3} + \dots + (q + 1)q^{k-m-1} \} \|f\|^p \\
&= \{ q^k + q^{k-2} + \dots + q^{k-2m} \} \|f\|^p \\
&\leq (q + 1)q^{k-1} \|f\|^p
\end{aligned}$$

showing that  $M_p^p(n, C_\phi f) \leq (q + 1)q^{k-1} \|f\|^p$  for  $n = 1, 2, \dots, k - 1$ . Next, for  $n = k$ , we find that

$$M_p^p(k, C_\phi f)$$

$$\begin{aligned}
&= \frac{1}{(q + 1)q^{k-1}} \sum_{|v|=k} |f(\phi(v))|^p \\
&\leq \frac{1}{(q + 1)q^{k-1}} \left\{ \sum_{|v|=2k} |f(v)|^p + \sum_{|v|=2k-2} |f(v)|^p + \dots + \sum_{|v|=2} |f(v)|^p + |f(o)|^p \right\} \\
&\leq \frac{1}{(q + 1)q^{k-1}} \{ (q + 1)q^{2k-1} + (q + 1)q^{2k-3} + \dots + (q + 1)q + 1 \} \|f\|^p \\
&= \left\{ q^k + q^{k-2} + \dots + q^{2-k} + \frac{1}{(q + 1)q^{k-1}} \right\} \|f\|^p \\
&\leq \{ q^k + q^{k-2} + \dots + q^{2-k} + q^{1-k} \} \|f\|^p \\
&\leq (q + 1)q^{k-1} \|f\|^p.
\end{aligned}$$

Finally, for each  $m \in \mathbb{N}_0$ ,

$$\begin{aligned}
M_p^p(m+k+1, C_\phi f) &= \frac{1}{(q+1)q^{m+k}} \sum_{|v|=m+k+1} |f(\phi(v))|^p \\
&\leq \frac{1}{(q+1)q^{m+k}} \left\{ \sum_{|v|=m+2k+1} |f(v)|^p + \sum_{|v|=m+2k-1} |f(v)|^p + \cdots + \sum_{|v|=2m+1} |f(v)|^p \right\} \\
&\leq \frac{1}{(q+1)q^{m+k}} \{ (q+1)q^{m+2k} + (q+1)q^{m+2k-2} + \cdots + (q+1)q^m \} \|f\|^p \\
&= \{q^k + q^{k-2} + \cdots + q^{-k}\} \|f\|^p \\
&\leq (q+1)q^{k-1} \|f\|^p.
\end{aligned}$$

The above discussion implies that

$$M_p^p(n, C_\phi f) \leq (q+1)q^{k-1} \|f\|^p \quad \text{for all } n \in \mathbb{N}_0$$

and thus,  $\|C_\phi\|^p \leq (q+1)q^{|\phi(o)|-1}$ . Other way inequality follows from Corollary 3 and the proof is complete.  $\square$

#### 4. COMPACT COMPOSITION OPERATORS

A bounded linear operator  $A$  from a normed linear space  $X$  to a normed linear space  $Y$  is said to be a *compact operator* if the image of closed unit ball  $\{Ax : \|x\|_X \leq 1\}$  has compact closure in  $Y$ .

In the classical case, for an analytic self map  $\phi$  of  $\mathbb{D}$ , the following statements are equivalent (see [12, Section 2.7 and Compactness Theorem, Chapter 10]):

- (a)  $C_\phi$  is compact on  $H^p$  for  $1 \leq p < \infty$ .
- (b)  $C_\phi$  is compact on  $H^2$ .
- (c)  $\lim_{|w| \rightarrow 1^-} \frac{N_\phi(w)}{\log \frac{1}{|w|}} = 0$ , where  $N_\phi$  is the Nevanlinna counting function of  $\phi$ .

Also,  $C_\phi$  is compact on  $H^\infty$  if and only if  $\sup\{|\phi(z)| : z \in \mathbb{D}\} < 1$  (see [12, Problem 10, Chapter 2]).

For the discrete setting, we now consider the compactness of composition operators on  $\mathbb{T}_p$  spaces. A self map  $\phi$  of  $(q+1)$ -homogeneous tree  $T$  is called a bounded map if  $\sup\{|\phi(v)| : v \in T\}$  is finite.

**Theorem 6.** *Every bounded self map  $\phi$  of  $T$  induces compact composition operator on  $\mathbb{T}_p$  for  $1 \leq p \leq \infty$ .*

*Proof.* Suppose  $\phi$  is a bounded self map of a  $(q+1)$ -homogeneous tree  $T$ . Then  $\text{Range}(\phi)$  is finite set, say,  $\text{Range}(\phi) = \{v_1, v_2, \dots, v_k\}$ . For each  $1 \leq i \leq k$ , denote by  $E_i$  for the pre-image of  $v_i$  under  $\phi$ . If  $\phi(v) = v_i$ , then  $f \circ \phi(v) = f(v_i)$  so that

$$f \circ \phi = f(v_1)\chi_{E_1} + f(v_2)\chi_{E_2} + \cdots + f(v_k)\chi_{E_k}$$



and  $\text{Range}(C_\phi) = \text{span}\{\chi_{E_1}, \chi_{E_2}, \dots, \chi_{E_k}\}$ . Thus,  $C_\phi$  is a finite rank operator and hence it is compact.  $\square$

**Theorem 7.** *If  $\phi$  is a self map of  $(q+1)$ -homogeneous tree  $T$ , then the following are equivalent:*

- (a)  $C_\phi$  is compact on  $\mathbb{T}_p$  for  $1 \leq p \leq \infty$ .
- (b)  $\|C_\phi f_n\| \rightarrow 0$  as  $n \rightarrow \infty$  whenever bounded sequence of functions  $\{f_n\}$  that converges to 0 pointwise.

*Proof.* (a)  $\Rightarrow$  (b): Assume that  $C_\phi$  is compact on  $\mathbb{T}_p$  and  $\{f_n\}$  is a bounded sequence in  $\mathbb{T}_p$  that converges to 0 pointwise. Suppose on the contrary that  $\|C_\phi(f_n)\| \not\rightarrow 0$  as  $n \rightarrow \infty$ . Then there exists a subsequence  $\{f_{n_j}\}$  and an  $\epsilon > 0$  such that  $\|C_\phi(f_{n_j})\| \geq \epsilon$  for all  $j$ . Denote  $\{f_{n_j}\}$  by  $\{g_j\}$ . Since  $C_\phi$  is compact, there is a subsequence  $\{g_{j_k}\}$  of  $\{g_j\}$  such that  $\{C_\phi(g_{j_k})\}$  converges to some function, say,  $g$ . It follows that  $\{C_\phi(g_{j_k})\}$  converges to  $g$  pointwise and  $g \equiv 0$  implying that  $\{C_\phi(g_{j_k})\}$  converges to 0 which is a contradiction to  $\|C_\phi(g_j)\| \geq \epsilon$  for all  $j$ . Hence,  $\|C_\phi(f_n)\| \rightarrow 0$  as  $n \rightarrow \infty$ .

(b)  $\Rightarrow$  (a): Conversely, suppose that case (b) holds. First let us consider the case  $1 \leq p < \infty$ . Let  $\{g_n\}$  be a sequence in unit ball of  $\mathbb{T}_p$ . By Lemma B, for each  $v \in T$ , the sequence  $\{g_n(v)\}$  is bounded. By the diagonalization process, there is a subsequence  $\{g_{nn}\}$  of  $\{g_n\}$  that converges pointwise to  $g$  (say). We see that, for each  $m \in \mathbb{N}_0$ ,

$$M_p^p(m, g) = \lim_{n \rightarrow \infty} \frac{1}{(q+1)q^{m-1}} \sum_{|v|=m} |g_{nn}(v)|^p \leq \limsup \|g_{nn}\|^p \leq 1$$

showing that  $g \in \mathbb{T}_p$  with  $\|g\| \leq 1$ . Consequently, if  $f_n = g_{nn} - g$ , then  $\{f_n\}$  converges to 0 pointwise and  $\|f_n\| \leq 2$ . By the assumption (b),  $\|C_\phi f_n\| \rightarrow 0$  as  $n \rightarrow \infty$  and thus,  $\{C_\phi(g_{nn})\}$  converges to  $C_\phi(g)$ . Hence  $C_\phi$  is compact on  $\mathbb{T}_p$ .

The proof for the case  $p = \infty$  is similar to the above.  $\square$

**Remark 1.** Since edge counting metric on  $T$  induces discrete topology, compact sets are only sets having finitely many elements. Thus convergence uniformly on compact subsets of  $T$  is equivalent to pointwise convergence. In view of this remark, Theorem 7 is a discrete analog of weak convergence theorem (see [12, section 2.4, p. 29]) in the classical case.

**Corollary 5.** *Let  $\phi$  be a self map of  $T$ . Then  $C_\phi$  is compact on  $\mathbb{T}_\infty$  if and only if  $\phi$  is a bounded self map of  $T$ .*

*Proof.* If  $\phi$  is a bounded self map of  $T$ , then  $C_\phi$  is compact, by Theorem 6. Conversely, suppose  $\phi$  is not a bounded map. Then, there exists a sequence of vertices  $\{v_k\}$  of  $T$  such that  $\phi(v_k) = w_k$  and  $|w_k| \rightarrow \infty$  as  $k \rightarrow \infty$ . Take  $f_k = \chi_{\{w_k\}}$  for each  $k \in \mathbb{N}$ . Then,  $\|f_k\|_\infty = 1$  for each  $k$  and  $\{f_k\}$  converges to 0 pointwise. Since  $C_\phi$  is compact,  $\|C_\phi(f_k)\|_\infty \rightarrow 0$  as  $k \rightarrow \infty$ , by Theorem 7. This is not possible, because  $\|C_\phi(f_k)\|_\infty = 1$  for each  $k \in \mathbb{N}$ , which can be observed from the definition of  $f_k$ . Hence  $\phi$  should be a bounded map.  $\square$

**Corollary 6.** *Let  $T$  be a  $(q+1)$ -homogeneous tree and  $1 \leq p < \infty$ . If  $C_\phi$  is compact on  $\mathbb{T}_p$ , then*

$$\sup_{n \in \mathbb{N}_0} \{q^{|w|-n} N_\phi(n, w)\} \rightarrow 0 \quad \text{as } |w| \rightarrow \infty.$$

*Proof.* As in the earlier situations, for each  $w \in T \setminus \{o\}$ , we let  $f_w = \{W(w)\chi_{\{w\}}\}^{\frac{1}{p}}$ . Then,  $\|f_w\| = 1$  for all  $w$  and, since  $f_w(v) = 0$  whenever  $|w| > n = |v|$ ,  $\{f_w\}$  converges to 0 pointwise. Since  $C_\phi$  is compact,  $\|C_\phi(f_w)\| \rightarrow 0$  as  $|w| \rightarrow \infty$ . However, we have already shown that

$$\|C_\phi f_w\|^p = \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_\phi(n, w) \right\} = \sup_{n \in \mathbb{N}_0} \{q^{|w|-n} N_\phi(n, w)\}$$

and the desired conclusion follows.  $\square$

**Remark 2.** For 2-homogeneous trees, Corollary 6 takes a simpler form: If  $C_\phi$  is compact on  $\mathbb{T}_p$ , then

$$\sup_{n \in \mathbb{N}_0} \{N_\phi(n, w)\} \rightarrow 0 \quad \text{as } |w| \rightarrow \infty.$$

This remark is helpful in the proof of Corollary 8.

**Corollary 7.** *If  $C_\phi$  is compact on  $\mathbb{T}_p$ , then  $|v| - |\phi(v)| \rightarrow \infty$  as  $|v| \rightarrow \infty$ .*

*Proof.* We will prove this result by contradiction. Suppose that  $|v| - |\phi(v)| \not\rightarrow \infty$  as  $|v| \rightarrow \infty$ . Then there exists a sequence of vertices  $\{v_k\}$  and an  $M > 0$  such that  $|v_k| - |\phi(v_k)| \leq M$  for all  $k$  which implies that  $|\phi(v_k)| \rightarrow \infty$  as  $k \rightarrow \infty$ . Since  $N_\phi(|v_k|, \phi(v_k)) \geq 1$  for all  $k$ , where  $N_\phi(n, w)$  is defined as in Section 3, we obtain that  $N_\phi(|v_k|, \phi(v_k)) q^{|\phi(v_k)| - |v_k|} \geq q^{-M}$  which yields that

$$\sup_{n \in \mathbb{N}_0} \{N_\phi(n, \phi(v_k)) q^{|\phi(v_k)| - n}\} \geq q^{-M} \quad \text{for all } k$$

and thus,

$$\sup_{n \in \mathbb{N}_0} \{q^{|w|-n} N_\phi(n, w)\} \not\rightarrow 0 \quad \text{as } |w| \rightarrow \infty$$

which gives that  $C_\phi$  is not compact, by Corollary 6. This contradiction completes the proof.  $\square$

**Corollary 8.** *Let  $T$  be a 2-homogeneous tree. Then  $C_\phi$  is compact on  $\mathbb{T}_p$  if and only if  $\phi$  is a bounded self map of  $T$ .*

*Proof.* Since every bounded self map  $\phi$  of  $T$  induces compact composition operator on  $\mathbb{T}_p$ , one way implication is hold. For the proof of the converse part, we suppose that  $\phi$  is not bounded. Then the range contains an infinite set, say  $\{w_1, w_2, \dots\}$ . For each  $k$ , choose  $v_k \in T$  such that  $\phi(v_k) = w_k$ . This gives  $N_\phi(|v_k|, w_k) \geq 1$  and thus  $\sup_{n \in \mathbb{N}_0} N_\phi(n, w_k) \geq 1$  for all  $k$ . It follows that  $\sup_{n \in \mathbb{N}_0} \{N_\phi(n, w)\} \not\rightarrow 0$  as  $|w| \rightarrow \infty$  and hence,  $C_\phi$  cannot be compact.  $\square$

**Remark 3.** It is worth to recall from [12, Chapter 3, p. 37] that if a “big-oh” condition describes a class of bounded operators, then the corresponding “little-oh” condition picks out the subclass of compact operators”. We have already shown that if  $\sum_{|v|=n} q^{|\phi(v)|} = O(q^n)$  then  $C_\phi$  is bounded on  $\mathbb{T}_p$ . So it is natural to ask whether  $\sum_{|v|=n} q^{|\phi(v)|} = o(q^n)$  guarantees the compactness of  $C_\phi$  on  $\mathbb{T}_p$ . Indeed, the answer is yes. Clearly the later observation is not useful because no self map  $\phi$  of  $T$  satisfies this condition. This is because  $\sum_{|v|=n} q^{|\phi(v)|} \geq \sum_{|v|=n} q^0 = (q+1)q^{n-1}$  and thus,  $\sum_{|v|=n} q^{|\phi(v)|} = o(q^n)$  is cannot be possible.

## 5. EXAMPLES

**Example 1.** For each  $n \in \mathbb{N}_0$ , choose the vertex  $v_n$  such that  $v_n \in D_n = \{v \in T : |v| = n\}$ . Define  $\phi_1(v) = v_n$  if  $|v| = n$ . Now, for  $q \neq 1$ , consider the function  $f$  defined by

$$f(v) = \begin{cases} \{(q+1)q^{n-1}\}^{\frac{1}{p}} & \text{if } v = v_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{elsewhere.} \end{cases}$$

Then  $f \in \mathbb{T}_p$ ,  $\|f\| = 1$  and for each  $m \in \mathbb{N}$ , we see that

$$M_p^p(m, C_{\phi_1} f) = \frac{1}{(q+1)q^{m-1}} \sum_{|v|=m} |f(v_m)|^p = (q+1)q^{m-1}$$

showing that  $\|C_{\phi_1} f\| = \sup_{m \in \mathbb{N}_0} M_p(m, C_{\phi_1} f)$  which is not finite for  $q \geq 2$ . This example shows that there are self maps of  $T$  which do not induce bounded composition operator on  $\mathbb{T}_p$  unlike the case of Hardy spaces on the unit disk.

**Example 2.** Consider the following self map  $\phi_2$  of  $T$  defined by

$$\phi_2(v) = \begin{cases} o & \text{if } v = o \\ v^- & \text{otherwise.} \end{cases}$$

where  $v^-$  denotes the parent of  $v$ . Then it follow easily that

$$M_p^p(0, C_{\phi_2} f) = M_p^p(0, f) \quad \text{and} \quad M_p^p(1, C_{\phi_2} f) = \frac{1}{(q+1)} \sum_{|v|=1} |f(o)|^p = M_p^p(0, f).$$

Finally, for  $n \geq 2$ , we have

$$\begin{aligned} M_p^p(n, C_{\phi_2} f) &= \frac{1}{(q+1)q^{n-1}} \sum_{|v|=n} |f(v^-)|^p \\ &= \frac{q}{(q+1)q^{n-1}} \sum_{|w|=n-1} |f(w)|^p \\ &= M_p^p(n-1, f) \end{aligned}$$

and thus,

$$\|C_{\phi_2} f\| = \sup_{m \in \mathbb{N}_0} M_p(m, C_{\phi_2} f) = \sup_{m \in \mathbb{N}_0} M_p(m, f) = \|f\|$$

showing that  $C_{\phi_2}$  is bounded on  $\mathbb{T}_p$ . On the other hand, since  $|v| - |\phi_2(v)| = 1$  for all  $|v| \geq 1$ , we have  $|v| - |\phi_2(v)| \not\rightarrow \infty$  as  $|v| \rightarrow \infty$ . Hence  $C_{\phi_2}$  is not compact, by

Corollary 7. This is an example of bounded composition operator on  $\mathbb{T}_p$  which is not compact.

**Remark 4.** Let  $\phi_3$  be a map on  $T$  such that  $\phi_3$  maps every vertex into any one of its child. Then, as in the case of  $C_{\phi_2}$ , it is easy to see that  $C_{\phi_3}$  is bounded but not compact. Moreover, it can be seen that, for each  $n \in \mathbb{N}$ ,  $(C_{\phi_2})^n$  and  $(C_{\phi_3})^n$  are also bounded but not compact.

**Example 3.** For each  $n \in \mathbb{N}_0$ , choose a vertex  $v_n$  such that  $|v_n| = n$ . Define a self map  $\phi_4$  by

$$\phi_4(v) = \begin{cases} v_k & \text{if } v = v_{2k} \text{ for some } k \in \mathbb{N} \\ o & \text{otherwise.} \end{cases}$$

Then we obtain that

$$M_p^p(0, C_{\phi_4}f) = |f(\phi_4(o))|^p = |f(o)|^p = M_p^p(0, f).$$

Next, for an odd natural number  $n$ , we see that

$$M_p^p(n, C_{\phi_4}f) = \frac{1}{(q+1)q^{n-1}} \sum_{|v|=n} |f(o)|^p = |f(o)|^p.$$

Finally, for an even natural number, say  $n = 2k$ , for some  $k \in \mathbb{N}$ , we find that

$$\begin{aligned} M_p^p(n, C_{\phi_4}f) &= \frac{1}{(q+1)q^{n-1}} \left\{ \sum_{\substack{|v|=n \\ v \neq v_{2k}}} |f(\phi_4(v))|^p + |f(\phi_4(v_{2k}))|^p \right\} \\ &\leq |f(o)|^p + \frac{|f(v_k)|^p}{(q+1)q^{n-1}}. \end{aligned}$$

Thus, by Lemma B, we have

$$\|C_{\phi_4}f\|^p \leq |f(o)|^p + \sup_{k \in \mathbb{N}} \left\{ \frac{|f(v_k)|^p}{(q+1)q^{2k-1}} \right\} \leq 2\|f\|^p$$

which shows that  $C_{\phi_4}$  is bounded on  $\mathbb{T}_p$ .

Suppose now that  $T$  is a  $(q+1)$ -homogeneous tree with  $q \geq 2$ . Let  $\{f_n\}$  be a sequence in the unit ball of  $\mathbb{T}_p$  which converges to 0 pointwise. Note that  $\left\{ \frac{|f(v_k)|^p}{(q+1)q^{2k-1}} \right\} \leq \frac{1}{q^k}$ , by Lemma B. We now claim that  $\|C_{\phi_4}f_n\|^p \rightarrow 0$  as  $n \rightarrow \infty$ .

Let  $\epsilon > 0$  be given. Then there exists a natural number  $N_1$  such that  $q^{-k} < \epsilon/2$  for all  $k \geq N_1$ . Consider the set  $S = \{v_1, v_2, \dots, v_{N_1}\}$ . Since  $\{f_n\}$  converges to 0 pointwise, we can choose a natural number  $N > N_1$  such that  $|f_n(o)|^p < \epsilon/2$  and  $|f_n(v)|^p < \epsilon/2$  for all  $v \in S$  and for all  $n \geq N$ . Thus,

$$\|C_{\phi_4}f_n\|^p \leq |f_n(o)|^p + \sup \left\{ \frac{\epsilon}{2}, \frac{1}{q^{N_1}}, \frac{1}{q^{N_1+1}}, \dots \right\} \leq \epsilon \text{ for all } n \geq N$$

which gives that  $\|C_{\phi_4}f_n\|^p \rightarrow 0$  as  $n \rightarrow \infty$  and hence  $C_{\phi_4}$  is compact on  $\mathbb{T}_p$ . This example shows that there are unbounded self maps of  $T$  which induces compact composition operators on  $\mathbb{T}_p$  for the case of  $(q+1)$ -homogeneous trees with  $q \geq 2$ .

We conclude the paper with a comparison in the case of  $q = 1$  and  $q > 1$ . For 2-homogeneous trees, circle of radius  $n$  has 2 vertices for all  $n \in \mathbb{N}$  whereas in the case of  $(q + 1)$ -homogeneous trees with  $q \geq 2$ , circle of radius  $n$  has  $(q + 1)q^{n-1}$  vertices. Due to this basic fact, we can expect a difference in operator theoretic point of view. The following table explains how composition operators on  $\mathbb{T}_p$  for  $1 \leq p < \infty$  differ in both the cases.

2-homogeneous tree	$(q + 1)$ -homogeneous tree with $q \geq 2$
Every self map $\phi$ of $T$ induces bounded composition operators on $\mathbb{T}_p$	There are self maps of $T$ which induces unbounded composition operators on $\mathbb{T}_p$ .
Only bounded self map of $T$ induces compact composition operators on $\mathbb{T}_p$	There are unbounded self maps of $T$ which induces compact operators on $\mathbb{T}_p$ .

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